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Anti Fuzzy Ideals of Lattice Wajsberg Algebras

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Abstract- In this paper, we introduce the definitions of an anti fuzzy *WI*-ideal and an anti fuzzy lattice ideal of lattice Wajsberg algebra. Further, we discuss the relationship between an anti fuzzy *WI*-ideal and an anti fuzzy lattice ideal. Also, we discuss some of its related properties.

Keywords: Wajsberg algebra, Lattice Wajsberg algebra, *WI*-ideal, Fuzzy *WI*-ideal, Fuzzy lattice ideal, Anti fuzzy *WI*-ideal, Anti fuzzy lattice ideal.

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1. INTRODUCTION

The concept of a fuzzy set was introduced by Zadeh[8] in 1965. Fuzziness occurs when the boundary of a piece of information is not clear-cut. Classical set theory allows the membership of the elements in the set in binary terms. Fuzzy set theory permits membership function valued in the interval [0, 1]. Mordchaj Wajsberg[7] introduced the concept of Wajsberg algebra in 1935 and studied by Font, Rodriguez and Torrens[3]. Also, they[3] defined lattice structure of Wajsberg algebra. Further, they[3] introduced the notion of an implicative filter of lattice Wajsberg algebra and discussed their properties. Basheer Ahamed and Ibrahim[1,2] introduced the definitions of fuzzy implicative and an anti fuzzy implicative filters of lattice Wajsberg algebra and investigated some properties. The authors[4,5] introduced the notions of *WI*-ideal, fuzzy *WI*-ideal, normal fuzzy *WI*-ideal of lattice Wajsberg algebra and studied some related properties.

In this paper, we introduce the definitions of an anti fuzzy *WI*-ideal and an anti fuzzy lattice ideal of lattice Wajsberg algebra. Further, we discuss the relationship between an anti fuzzy *WI*-ideal and an anti fuzzy lattice ideal in lattice Wajsberg algebra as well as in lattice *H*-Wajsberg algebra. Moreover, we investigate some of its related properties.

2. PRELIMINARIES

In this section, we recall some basic notions and their properties that are helpful to develop our main results.

Definition 2.1[3]. Let (A, $\rightarrow$, $*$, 1) be an algebra with quasi complement "*" and a binary operation "$\rightarrow$" is called a Wajsberg algebra if and only if it satisfies the following axioms for all $x, y, z \in A$,

(i) $1 \rightarrow x = x$

(ii) $(x \rightarrow y) \rightarrow ((y \rightarrow z) \rightarrow (x \rightarrow z)) = 1$

(iii) $(x \rightarrow y) \rightarrow y = (y \rightarrow x) \rightarrow x$

(iv) $(x^* \rightarrow y^*) \rightarrow (y \rightarrow x) = 1$.

Proposition 2.2[3]. A Wajsberg algebra (A, $\rightarrow$, $*$, 1) satisfies the following axioms for all $x, y, z \in A$,

(i) $x \rightarrow x = 1$

(ii) If $(x \rightarrow y) = (y \rightarrow x) = 1$ then $x = y$

(iii) $x \rightarrow 1 = 1$

(iv) $(x \rightarrow (y \rightarrow x)) = 1$

(v) If $(x \rightarrow y) = (y \rightarrow z) = 1$ then $x \rightarrow z = 1$

(vi) $(x \rightarrow y) \rightarrow ((z \rightarrow x) \rightarrow (z \rightarrow y)) = 1$

(vii) $x \rightarrow (y \rightarrow z) = y \rightarrow (x \rightarrow z)$

(viii) $x \rightarrow 0 = x \rightarrow 1^* = x^*$

(ix) $(x^*)^* = x$

(x) $(x^* \rightarrow y^*) = y \rightarrow x$.

Definition 2.3[3]. Wajsberg algebra A is called a lattice Wajsberg algebra, if it satisfies the following conditions for all $x, y \in A$,

(i) The Partial ordering $\leq$ on a lattice Wajsberg algebra A, such that $x \leq y$ if and only if $x \rightarrow y = 1$

(ii) $(x \vee y) = (x \rightarrow y) \rightarrow y$

(iii) $(x \wedge y) = ((x^* \rightarrow y^*) \rightarrow y^*)^*$.

Thus (A, $\vee$, $\wedge$, $*$, 0, 1) is a lattice Wajsberg algebra with lower bound 0 and upper bound 1.

Proposition 2.4[3]. A Wajsberg algebra $(A, \to, *, 1)$ satisfies the following axioms for all $x, y, z \in A$,

(i) If $x \leq y$ then $x \to z \geq y \to z$ and $z \to x \leq z \to y$

(ii) $x \leq y \to z$ if and only if $y \leq x \to z$

(iii) $(x \vee y)^* = (x^* \wedge y^*)$

(iv) $(x \wedge y)^* = (x^* \vee y^*)$

(v) $(x \vee y) \to z = (x \to z) \wedge (y \to z)$

(vi) $x \to (y \wedge z) = (x \to y) \wedge (x \to z)$

(vii) $(x \to y) \vee (y \to x) = 1$

(viii) $x \to (y \vee z) = (x \to y) \vee (x \to z)$

(ix) $(x \wedge y) \to z = (x \to z) \vee (y \to z)$

(x) $(x \wedge y) \vee z = (x \vee z) \wedge (y \vee z)$

(xi) $(x \wedge y) \to z = (x \to y) \to (x \to z)$.

Definition 2.5[4]. The lattice Wajsberg algebra A is called a lattice H-Wajsberg algebra, if it satisfies $x \vee y \vee ((x \wedge y) \to z) = 1$ for all $x, y, z \in A$.
In a lattice H-Wajsberg algebra A, the following hold,

(i) $x \to (x \to y) = (x \to y)$

(ii) $x \to (y \to z) = (x \to y) \to (x \to z)$.

Definition 2.6[3]. Let L be a lattice. An ideal I of L is a nonempty subset of L is called a lattice ideal, if it satisfies the following axioms for all $x, y \in I$,

(i) $x \in I, y \in L$ and $y \leq x$ imply $y \in I$

(ii) $x, y \in I$ implies $x \vee y \in I$.

Definition 2.7[4]. Let A be a lattice Wajsberg algebra. Let I be a nonempty subset of A, then I is called WI-ideal of lattice Wajsberg algebra A satisfies,

(i) $0 \in I$

(ii) $(x \to y)^* \in I$ and $y \in I$ imply $x \in I$ for all $x, y \in A$.

Definition 2.8[8]. Let A be a set. A function $\mu : A \to [0,1]$ is called a fuzzy subset on A, for each $x \in A$, the value of $\mu(x)$ describes a degree of membership of x in μ.

Definition 2.9[8]. Let μ be a fuzzy set in a set A. Then for $t \in [0,1]$, the set $\mu_t = \{x \in A / \mu(x) \geq t\}$ is called level subset of μ.

Definition 2.10[8]. Let μ be a fuzzy subset of a set A. Then for $t \in [0,1]$, the set $\mu^t = \{x \in A / \mu(x) \leq t\}$ is called the lower t-level cut of μ.

Definition 2.11[5]. Let A be a lattice Wajsberg algebra. A fuzzy subset μ of A is called a fuzzy WI-ideal of A if for any $x, y \in A$,

(i) $\mu(0) \geq \mu(x)$

(ii) $\mu(x) \geq \min\{\mu((x \to y)^*), \mu(y)\}$.

Definition 2.12[5]. A fuzzy subset μ of a lattice Wajsberg algebra A is called fuzzy lattice ideal if for any $x, y \in A$.

(i) If $y \leq x$, then $\mu(y) \geq \mu(x)$

(ii) $\mu(x \vee y) \geq \min\{\mu(x), \mu(y)\}$.

Definition 2.13[5]. Let $\mathcal{F}(A)$ be the set of all anti fuzzy subset of lattice Wajsberg algebra A and $\mu \in \mathcal{F}(A)$, we define the set as $A_\mu = \{x \in A / \mu(x) = \mu(0)\}$.

3. PROPERTIES OF ANTI FUZZY *WI*-IDEALS

In this section, we define an anti fuzzy WI-ideal in lattice Wajsberg algebra and obtain some useful results with illustrations.

Definition 3.1. A fuzzy subset μ of lattice Wajsberg algebra A is called an anti fuzzy WI-ideal of A if for any $x, y \in A$,

(i) $\mu(0) \leq \mu(x)$

(ii) $\mu(x) \leq \max\{\mu((x \to y)^*), \mu(y)\}$.

Example 3.2. Let $A = \{0, e, f, g, h, 1\}$ be a set with partial ordering. Define a quasi complement "$*$" and a binary operation "$\to$" on A as in tables (3.1) and (3.2).

x	x^*
0	1
e	g
f	f
g	e
h	0
1	0

Table: 3.1 Complement

$\to$	0	e	f	g	h	1
0	1	1	1	1	1	1
e	g	1	1	1	1	1
f	f	f	1	1	1	1
g	e	e	f	1	1	1
h	0	e	f	g	1	1
1	0	e	f	g	h	1

Table: 3.2 Implication

Define $\vee$ and $\wedge$ operations on A as follow,

$(x \vee y) = (x \to y) \to y$,

$(x \wedge y) = ((x^* \to y^*) \to y^*)^*$ for all $x, y \in A$.

Then, *A* is a lattice Wajsberg algebra.

Consider the fuzzy subset μ on *A* as,

$$\mu(x) = \begin{cases} 0.1 & if \quad x = 0 \\ 0.3 & otherwise \end{cases} \quad \text{for all } x \in A$$

Then, we have μ is an anti fuzzy *WI*-ideal of lattice Wajsberg algebra *A*.

Proposition 3.3. Every anti fuzzy *WI*-ideal μ of a lattice Wajsberg algebra *A* is order preserving.

Proof. Let μ be an anti fuzzy *WI*-ideal of *A*.

For any $x, y \in A$ and $x \le y$ then $(x \to y)^* = 1^* = 0$

and so, $\mu(x) \le \max\{\mu((x \to y)^*),\ \mu(y)\}$

$= \max\{\mu(0),\ \mu(y)\}$

$= \mu(y)$

Thus, $\mu(x) \le \mu(y)$

Hence, μ is order preserving. ■

Definition 3.4. A fuzzy subset μ of a lattice Wajsberg algebra *A* is called an anti fuzzy lattice ideal if for any $x, y \in A$,

(i) If $y \le x$, then $\mu(y) \le \mu(x)$

(ii) $\mu(x \vee y) \le \max\{\mu(x), \mu(y)\}$.

Proposition 3.5. Every anti fuzzy *WI*-ideal of lattice Wajsberg algebra *A* is an anti fuzzy lattice ideal.

Proof. Let μ be an anti fuzzy *WI*-ideal of *A*.

From Proposition 3.3, $\mu(y) \le \mu(x)$ if $y \le x$

By $((x \vee y) \to y)^* = ((x \to y) \wedge (y \to y))^*$

$= (x \to y)^*$

$\le x$

We get, $\mu(x \vee y) \le \max\{\mu((x \vee y) \to y)^*), \mu(y)\}$

$\le \max\{\mu(x), \mu(y)\}$.

Thus, $\mu(x \vee y) \le \max\{\mu(x), \mu(y)\}$ ■

The following example shows that the converse of Proposition 3.5 is not true.

Example 3.6. Let $A = \{0, l, m, n, 1\}$ be a set with partial ordering. Define a quasi complement "$*$" and a binary operation "$\to$" on *A* as in tables (3.3) and (3.4).

x	x^*
0	1
l	n
m	m
n	l
1	0

Table: 3.3
Implication

$\to$	0	l	m	n	1
0	1	1	1	1	1
l	n	1	1	1	1
m	m	n	1	1	1
n	l	m	n	1	1
1	0	l	m	n	1

Tables: 3.4
Complement

Define $\vee$ and $\wedge$ operations on *A* as follow,

$(x \vee y) = (x \to y) \to y$,

$(x \wedge y) = ((x^* \to y^*) \to y^*)^*$ for all $x, y \in A$.

Then, *A* is a lattice Wajsberg algebra.

Consider the fuzzy subset μ on *A* as,

$$\mu(x) = \begin{cases} 0.2 & if\ x \in \{0, n, 1\}\ for\ all\ x \in A \\ 0.7 & if\ x \in \{l, m\} \quad for\ all\ x \in A \end{cases}$$

Then, μ is an anti fuzzy lattice ideal of *A*, but not an anti fuzzy *WI*-ideal for

$$\mu(m) \nleq \max\{\mu((m \to n)^*),\ \mu(n)\}.$$

Proposition 3.7. Every anti fuzzy lattice ideal is an anti fuzzy *WI*-ideal in lattice *H*-Wajsberg algebra *A*.

Proof. Let μ be an anti fuzzy lattice ideal of lattice *H*-Wajsberg algebra *A*.

If $y \le x$, then $\mu(y) \le \mu(x)$ and $\mu(x \vee y) \le \max\{\mu(x), \mu(y)\}$ for all $x, y \in A$.

Since $0 \le x$, it follows that $\mu(0) \le \mu(x)$ for all $x \in A$

[From (i) of Definition 3.4]

Let $x, y \in A$ we have,

$\mu(x) \le \mu(x \vee y)$

$= \mu\ (y \vee (x^* \vee y^*)^*)$

$= \mu(y \vee (x \to y)^*)$

$\le \max\{\mu(y),\ \mu(x \to y)^*\}$

Hence, μ is an anti fuzzy *WI*-ideal. ■

Proposition 3.8. A fuzzy subset μ of a lattice Wajsberg algebra *A* is an anti fuzzy *WI*-ideal if and only if μ_t is a *WI*-ideal when $\mu_t \ne \phi$, $t \in [0,1]$.

Proof. Let μ be an anti fuzzy *WI*-ideal of *A* and let $t \in [0,1]$ such that $\mu_t \ne \phi$.Clearly, $0 \in \mu_t$.

Suppose $x, y \in A$, $(x \to y)^* \in \mu_t$ and $y \in \mu_t$.
Then, $\mu((x \to y)^*) \leq t$ and $\mu(y) \leq t$.
It follows that, $\mu(x) \leq \max\{\mu(x \to y)^*, \mu(y)\} \leq t$.
So that, $x \in \mu_t$. Hence, μ_t is a *WI*-ideal of *A*.

Conversely, suppose μ_t $(t \in [0,1])$ is a *WI*-ideal of *A*. When $\mu_t \neq \phi$ for any $x \in A$, $x \in \mu_{\mu(x)}$, it follows that $\mu_{\mu(x)}$ is a *WI*-ideal of *A* and hence $0 \in \mu_{\mu(x)}$ that is $\mu(0) \leq \mu(x)$ for any $x, y \in A$.
Let $t = \max\{\mu(x \to y)^*, \mu(y)\}$, it follows that μ_t is a *WI*-ideal and $(x \to y)^* \in \mu_t$, $y \in \mu_t$, this implies that $x \in \mu_t$ and $\mu(x) \leq t = \max\{\mu(x \to y)^*, \mu(y)\}$. ■

Proposition 3.9. Let *A* be a lattice Wajsberg algebra. If μ and σ are non-empty anti fuzzy *WI*-ideals of *A* such that for any $x, y \in A$, $\mu(x) \geq \mu(y)$ if and only if $\sigma(x) \geq \sigma(y)$ then $\mu \circ \sigma$ is also a fuzzy *WI*-ideal of *A* and $A_{\mu \circ \sigma} = A_\mu \cap A_\sigma$ where $(\mu \circ \sigma)(x) = \mu(x)\sigma(x)$ for any $x \in A$.

***Proof*.** It is easy to prove that, $\mu \circ \sigma$ is a *WI*-ideal of *A* and $A_{\mu \circ \sigma} \supseteq A_\mu \cap A_\sigma$
Let $x \in A_{\mu \circ \sigma}$. Then $(\mu \circ \sigma)(x) = \mu \circ \sigma(0)$

$$\mu(x)\sigma(x) = \mu(0)\sigma(0)$$

Hence, $\mu(x) \neq 0$ and $\sigma(x) \neq 0$.
If $\mu(x) > \mu(0)$, then $\mu(x)\sigma(x) > \mu(0)\sigma(0)$.
This is a contradiction. Similarly, it is also a contradiction when $\sigma(x) > \sigma(0)$.
Hence, $\mu(x) = \mu(0)$ and $\sigma(x) = \sigma(0)$. That is $x \in A_\mu \cap A_\sigma$. ■

4. CONCLUSION

In the present paper, we have introduced the notions of an anti fuzzy *WI*-ideal and an anti fuzzy lattice ideal of lattice Wajsberg algebra. Further, we have shown that every anti fuzzy *WI*-ideal of a lattice Wajsberg algebra is order preserving. Also, we have obtained some of its properties.

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